ERRORS IN TWO DIMENSIONAL GAUSSIAN FITS

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Abstract

Two-dimensional elliptical Gaussian fits are used in astronomy for accurate measurements of source parameters such as central position, peak flux density and angular size. Condon [1] has provided an analysis of error propagation for two-dimensional elliptical Gaussian fits in coordinates along the major and minor axes. This memo concerns the implementation of error estimates within the fitting task JMFIT. The problem of error estimates can be considered more generally in the RA-DEC coordinate system for an ellipse of arbitrary position angle. This analysis is presented and a difference at the different ratio of the beam and the fitted gausian size is discussed. Three sets of formulae for the errors have been obtained: 1) the case of the point source (the beam size $\simeq$ the gaussian size); 2) the case of a resolved source (the beam size $\ll$ the gaussian size); and 3) The intermediate case.

1 Introduction

The fitting of two-dimensional elliptical Gaussians is common in astronomy and the appropriate error analysis for the six fitted parameters is very important. Condon [1] has provided an analysis of error propagation for two-dimensional elliptical Gaussian fits in coordinates along the major and minor axes. Such a special coordinate system does not allow the determination of all six parameter errors for an ellipse of arbitrary position angle. Using the linear transformation between two cartesian coordinate systems, the following expression can be used however:

$$\begin{aligned} DX_1 &= DX \cdot \sin^2\varphi \ + \ DY \cdot \cos^2\varphi \\ DY_1 &= DX \cdot \cos^2\varphi \ + \ DY \cdot \sin^2\varphi \end{aligned} \tag{1}$$

where DX_1, DY_1 are variances of the central position at the RA-DEC coordinate system;
DX, DY are variances of the central position at the coordinates along the major and minor axes;
φ is position angle of the major axis

Such a formula is implemented in the AIPS task JMFIT. It merits further investigation however, because, a dependence on the variance of position angle may be expected. This problem can be considered more generally by performing an analysis of error propagation for two-dimensional Gaussian fits in an RA-DEC coordinate system.
Many formulae in this memo were taken from the Condon [1] paper.

2 Error Propagation in an RA-DEC Coordinate System: Independent Samples.

Condon [1] obtained the following matrix using coordinates along the major and minor axes:

$$M = \frac{\pi A}{2} \begin{array}{c} \left(\begin{array}{cccccc} \frac{A\sigma_y}{\sigma_x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A\sigma_x}{\sigma_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{A\sigma_x\sigma_y}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3A\sigma_y}{2\sigma_x} & \frac{A}{2} & \sigma_y \\ 0 & 0 & 0 & \frac{A}{2} & \frac{3A\sigma_x}{2\sigma_y} & \sigma_x \\ 0 & 0 & 0 & \sigma_y & \sigma_x & \frac{2\sigma_x\sigma_y}{A} \end{array} \right) \end{array} \begin{array}{c} X \\ Y \\ \beta \\ W_x \\ W_y \\ A \end{array} \tag{2}$$

where the right-hand column denotes the sequence of the fitted parameters;
X,Y are coordinates of the ellipse center;
β is a parameter determining the position angle of the major axis;
W_x, W_y are the width of the gaussian at the major and minor axis direction;
A is the maximum value of the gaussian;

The widths of the Gaussian and location of its center are measured throughout this memo in pixels. The parameter sequence has been rearranged for the matrix to have diagonal type to simplify evaluation of the inverse matrix. In an RA-DEC coordinate system, matrix (2) takes the form:

$$M = \frac{\pi A}{2} \left(\begin{array}{cccccc} CS & S2F & 0 & 0 & 0 & 0 \\ S2F & SC & 0 & 0 & 0 & 0 \\ 0 & 0 & DF^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3A}{2\alpha}\frac{\theta_m}{\theta_M} & \frac{A}{2\alpha} & \frac{\theta_m}{\alpha} \\ 0 & 0 & 0 & \frac{A}{2\alpha} & \frac{3A}{2\alpha}\frac{\theta_M}{\theta_m} & \frac{\theta_M}{\alpha} \\ 0 & 0 & 0 & \frac{\theta_m}{\alpha} & \frac{\theta_M}{\alpha} & \frac{2\theta_M\theta_m}{\alpha A} \end{array} \right) \begin{array}{c} X \\ Y \\ \varphi \\ \theta_M \\ \theta_m \\ A \end{array} \tag{3}$$

where the right column denotes the sequence of the fitted parameters;
X,Y are coordinates of the ellipse center;
φ is the position angle of the major axis;
θ_M, θ_m are the length of the major and minor ellipse axes at the half amplitude point;
A is the maximum value of the Gaussian;
$\alpha = 8 \ln 2$
$CS = A\left(\frac{\theta_m}{\theta_M}\sin^2\varphi + \frac{\theta_M}{\theta_m}\cos^2\varphi\right)$
$SC = A\left(\frac{\theta_m}{\theta_M}\cos^2\varphi + \frac{\theta_M}{\theta_m}\sin^2\varphi\right)$
$S2F = \frac{A}{2}\frac{\theta_M^2-\theta_m^2}{\theta_M\theta_m}\sin 2\varphi$
$DF^{-1} = \frac{A}{2\alpha}\frac{\left(\theta_M^2-\theta_m^2\right)^2}{\theta_M\theta_m}$

Matrices (2) and (3) differ by two additional nonzero terms which demonstrate the correlation of the center coordinate errors, if the major and minor axes do not coincide with the coordinate axes. The inverse matrix M^{-1} (the error matrix) can be evaluated in this case as a combination of three inverse matrices: a 2x2 top-left matrix, a 1x1 middle matrix and a 3x3 bottom-right matrix. The resulting error matrix takes the form:

$$M^{-1} = \frac{2}{\pi A} \left(\begin{array}{cccccc} \frac{SC}{D} & -\frac{S2F}{D} & 0 & 0 & 0 & 0 \\ -\frac{S2F}{D} & \frac{CS}{D} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{DF^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{A}\frac{\theta_M}{\theta_m} & 0 & -\frac{\alpha}{2\theta_m} \\ 0 & 0 & 0 & 0 & \frac{\alpha}{A}\frac{\theta_m}{\theta_M} & -\frac{\alpha}{\theta_M} \\ 0 & 0 & 0 & -\frac{\alpha}{2\theta_m} & -\frac{\alpha}{2\theta_M} & \frac{A\alpha}{\theta_M\theta_m} \end{array} \right) \begin{array}{c} X \\ Y \\ \varphi \\ \theta_M \\ \theta_m \\ A \end{array} \tag{4}$$

where

$D = CS \cdot SC \; - \; S2F^2$ is the determinant of the top-left 2x2 matrix.

In spite of the fact that the initial expression for D includes a rather complicated dependence on position angle φ, after simplification this dependence disappears and the determinant is given by the very simple formula: $D = A^2$!!! If samples of amplitude measurements are independent and have the same rms σ, the variance of any differentiable function $\mathbf{F}$ of the six fitted Gaussian parameters is given by the following function of σ and elements of matrix M^{-1}:

$$VAR(F) = \sigma^2 \sum_{i=1}^{6}\sum_{j=1}^{6} M_{ij}^{-1} \frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j} \tag{5}$$

In particular, the variance of the fitted parameters is determined exclusively by the diagonal elements of the matrix :

$$VAR(p_i) = \sigma^2 M_{ii}^{-1} \tag{6}$$

$$\begin{aligned} \frac{2}{\rho^2} &= \frac{2\alpha}{\pi\theta_M\theta_m}\frac{\sigma^2}{A^2} = \frac{VAR(A)}{A^2} = \frac{VAR(\theta_m)}{\theta_m^2} = \frac{VAR(\theta_M)}{\theta_M^2} = \frac{VAR(X)}{\theta_M^2\sin^2\varphi + \theta_m^2\cos^2\varphi}\alpha = \\ &= \frac{VAR(Y)}{\theta_m^2\sin^2\varphi + \theta_M^2\cos^2\varphi}\alpha = \frac{VAR(\varphi)}{2}\left(\frac{\theta_M^2 - \theta_m^2}{\theta_m\theta_M}\right)^2 \end{aligned} \tag{7}$$

The number ρ specified the overall signal-to-noise ratio of the Gaussian was implemented by Condon [1] and is determined by the following equation in the case of independent samples:

$$\rho^2 = \frac{\pi}{\alpha}\frac{A^2}{\sigma^2}\theta_M\theta_m \tag{8}$$

The integrated amplitude is defined by:

$$I = \frac{2\pi}{\alpha}\theta_M\theta_m A \tag{9}$$

Its variance inferred from equation (5) equals:

$$\begin{aligned} VAR(I) &= \sigma^2[M_{66}^{-1}\left(\frac{\partial I}{\partial A}\right)^2 + M_{55}^{-1}\left(\frac{\partial I}{\partial\theta_m}\right)^2 + M_{44}^{-1}\left(\frac{\partial I}{\partial\theta_M}\right)^2 + \\ &+ 2M_{56}^{-1}\left(\frac{\partial I}{\partial A}\right)\left(\frac{\partial I}{\partial\theta_m}\right) + 2M_{46}^{-1}\left(\frac{\partial I}{\partial A}\right)\left(\frac{\partial I}{\partial\theta_M}\right)] \end{aligned} \tag{10}$$

The fourth and fifth items of the sum in equation (10) are negative because errors of maximum value and width of the ellipse are anti-correlated (as seen by examining the corresponding coefficients of the matrix equation (4). Forming the sum in equation (10) shows that the absolute value of the second and third items are equal to the absolute value of the fourth and fifth items respectively. So they cancel and the variance of the integrated amplitude is given by the first term only:

$$\frac{VAR(I)}{I^2} = \frac{VAR(A)}{A^2} \tag{11}$$

This result is valid only for independent samples. The general case is more complicated and is partially analyzed in the following section.

3 Correlated Noise.

Generally speaking, there is no image having independent pixel noise, because any image is the result of convolution of an actual image with the synthesized beam. Mathematically it is extremely difficult to provide a full analysis of error propagation for dependent pixel noise if the beam and fitted component have arbitrary size and orientation. Condon [1] has analyzed the case of circular beam. Let's consider the two limiting cases: 1) beam size is much bigger original source size i.e. image coincides with beam; 2) beam size is much smaller than original source size. For these two limiting cases Condon's analysis can be expanded to a general form of the beam by substituting the square of the beam diameter θ_N^2 by the product of the major and minor axes of the beam $\theta_{bM}\theta_{bm}$. Now let's look at the two limiting cases separately.

Expanded source. Beam size is much smaller than original source size.
Following Condon [1] a new expression for ρ has to be used instead of equation (7):

$$\frac{2}{\rho^2} = 8\frac{\sigma^2}{A^2}\frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} \tag{12}$$

Equation (7) can be used to determined errors of the six fitted parameters but using ρ taken from equation (12). The error in the integrated amplitude (11) is transformed to:

$$\frac{VAR(I)}{I^2} = \frac{VAR(A)}{A^2} + \frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m}\left[\frac{VAR(\theta_M)}{\theta_M^2} + \frac{VAR(\theta_m)}{\theta_m^2}\right] \tag{13}$$

Point source. Image coincides with beam.
Parameter ρ is given in this case by the following expression:

$$\frac{2}{\rho^2} = \frac{\sigma^2}{A^2} \tag{14}$$

The six fitted parameters can be obtained from (7) using equation (14) for ρ. In particular $VAR(A) = \sigma^2$. The error in integrated amplitude now takes the form:

$$\frac{VAR(I)}{I^2} = \frac{VAR(A)}{A^2} + \frac{VAR(\theta_M)}{\theta_M^2} + \frac{VAR(\theta_m)}{\theta_m^2} \tag{15}$$

Intermediate case. The source is partially resolved
This case was analyzed by Condon [1] for a circular beam. For a general beam this problem is very difficult. A reasonable approach to the problem can be interpolation of expression for $\frac{2}{\rho^2}$ using equation (7) and (13):

$$\frac{2}{\rho^2} = \frac{VAR(A)}{A^2} = \begin{cases} 8\frac{\sigma^2}{A^2}\frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} & \text{if } 0.0 \le \frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} < 0.1 \\ \frac{\sigma^2}{A^2}\left[0.8 + \frac{1}{4}\left(\frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} - 0.1\right)\right] & \text{if } 0.1 \le \frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} < 0.9 \\ \frac{\sigma^2}{A^2} & \text{if } 0.9 \le \frac{\theta_{bM}\theta_{bm}}{\theta_M\theta_m} < 1.0 \end{cases} \tag{16}$$

4 Conclusion. What Has Been Implemented Within JMFIT?

Several modifications in the errors calculation within JMFIT were implemented as a result of this memo:
1. The old version of JMFIT had one set of formulae independent of the ratio of the beam size to the image size. The new version (15OCT96) analyses this ratio and uses three sets of the formulae as given in equation (16) above.
2. The old version of JMFIT provided two sets of errors based on the rms obtained from the residual

map and the rms from the data itself, which caused confusion. Only one set of errors based on the rms obtained from the data itself has been retained. A new adverb has been added to allow the user to force a pre-determined rms of samples to be used in the error calculations.

3. The variance of the maximum value A can be halved by fixing the values of width and position angle of the gaussian ellipse. It follows from the anti-correlation of the errors of the ellipse size and the maximum value [1]. This reduction of the error of the maximum value has been implemented in JMFIT when adverb GWIDTH is used to fix the major and minor axes and position angle leaving only three free parameters: maximum value and X, Y coordinates of the ellipse center.

4. A typographical error in the calculation of the position angle error has been corrected.

References

[1] J.J. Condon, Preprint of National Radio Astronomy Observatory May, 1996