AIPS Memo 46

Additional Non-linear Coordinates in $\mathcal{AIPS}$
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Eric W. Greisen*

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Abstract

Four additional non-linear coordinate types have been added to $\mathcal{AIPS}$. These are the stereographic projective geometry and the non-projective Aitoff, "global-sinusoidal" and Mercator geometries appropriate to the display of very large fields.

Added January 1993: It is the author's current opinion that the generalizations of these coordinates to non-null reference values, given below, is not the most desirable. A new and more functional generalization is explicated in the 1993 proposed FITS conventions. Substantial changes from this memo to the 1993 proposal are footnoted where appropriate.

1 Introduction

In $\mathcal{AIPS}$ Memo No. 27 (1983), the basic non-linear coordinates used in $\mathcal{AIPS}$ were described. These included three projective, tangent-plane geometries called SIN, TAN, and ARC, descriptive of the form of the projection. An additional geometry using a projection to a plane tangent to the pole was also described. In the `15JUL86` release of $\mathcal{AIPS}$, four additional non-linear geometries are supported. These are the stereographic projective geometry and the non-projective Aitoff, "global-sinusoidal" and Mercator geometries appropriate to the display of very large fields. This memorandum describes the algebra and $\mathcal{AIPS}$ parameters used in this implementation.

2 The Coordinates

Like the SIN, TAN, and ARC geometries, the stereographic geometry is a true projection to a plane tangent to the celestial sphere at the reference pixel. It is a tangent projection from the opposite side of the celestial sphere; the TAN projection is a projection from the center of the sphere. The stereographic projection has the attribute that celestial circles remain circles in the projected image and is also useful for displaying polar regions and large fields.

The Aitoff system has been used with IRAS data and is intended to be one of the primary systems used for large-field imagery from the Space Telescope. The global-sinusoidal coordinate system is an obvious system in astronomy and is usually what people mean when they assert that their coordinates are α–δ. Both of these projections have the "equal-area" property; this means that a unit area on the sky (*e.g.*, one square spherical arc second) anywhere in the field of view projects to a constant number of pixels in the digital image. None of the four projections described in $\mathcal{AIPS}$ Memo No. 27 has this property, nor does the stereographic and Mercator projections described below. Equal-area projections conserve both the

*National Radio Astronomy Observatory

numerical values of surface brightness and fluxes integrated over areas, making them suitable for most forms of photometric analysis. However, they distort angular relationships and, hence, shapes, especially near the poles. The global sinusoidal geometry has the advantage of simplicity, but the distortions are less severe in the Aitoff geometry.

The Mercator system is sometimes used in planetary studies and was developed as an aid to navigation. It has the property that "rhumb" lines are straight, where a rhumb line is the trajectory produced by maintaining a constant compass reading as latitude changes. In general, arcs of great circles are more interesting in astronomy and appear as straight lines in the gnomic projection (called TAN in *AIPS*). The four new systems will be referred to here by their initials — STG, GLS, AIT, and MER, respectively.

In a plane, the position of a point (x, y) with respect to the coordinate reference point in an arbitrary linear system may be represented as

$$\begin{aligned} x &= L\cos\rho + M\sin\rho \\ y &= M\cos\rho - L\sin\rho\,, \end{aligned} \tag{1}$$

or

$$\begin{aligned} L &= x\cos\rho - y\sin\rho \\ M &= y\cos\rho + x\sin\rho\,, \end{aligned} \tag{2}$$

where ρ is a rotation, L is the direction cosine parallel to latitude at the reference pixel, and M is the direction cosine parallel to longitude at the reference pixel. Both the (x, y) and (L, M) systems are simple linear, perpendicular systems. If we represent longitude and latitude with the symbols α and δ, the fun arises in solving the four problems: (i) given α, δ find x, y; (ii) given x, y find α, δ; (iii) given x, δ find α, y; and (iv) given α, y find x, δ. The solutions to these problems for the three new coordinate systems are given in the sections below. In the derivations, I will use the definitions $\Delta\alpha \equiv \alpha - \alpha_0$ and $\Delta\delta \equiv \delta - \delta_0$ for simplicity and the subscript 0 to refer to quantities evaluated at the reference pixel.

2.1 STG geometry

2.1.1 Find x, y from α, δ

The STG geometry requires the usual basic formulæ from spherical triangles:

$$\cos\theta = \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha \tag{3}$$

$$\sin\theta\sin\phi = \cos\delta\sin\Delta\alpha \tag{4}$$

$$\sin\theta\cos\phi = \sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha\,. \tag{5}$$

A simple extension of Figure 1 in *AIPS* Memo No. 27 followed by simple trigonometry yields the projections

$$\begin{aligned} L &= 2\,\frac{\sin\theta}{1+\cos\theta}\sin\phi \\ M &= 2\,\frac{\sin\theta}{1+\cos\theta}\cos\phi \end{aligned} \tag{6}$$

or, substituting equations 3, 4, and 5,

$$\begin{aligned} L &= 2\,\frac{\cos\delta\sin\Delta\alpha}{1+\sin\delta\sin\delta_0+\cos\delta\cos\delta_0\cos\Delta\alpha} \\ M &= 2\,\frac{\sin\delta\cos\delta_0-\cos\delta\sin\delta_0\cos\Delta\alpha}{1+\sin\delta\sin\delta_0+\cos\delta\cos\delta_0\cos\Delta\alpha}\,. \end{aligned} \tag{7}$$

Then equations 1 determine x and y.

2.1.2 Find α, δ from x, y

Equations 2 yield L and M from x and y. Then, using equations 6,

$$L^2 + M^2 = \frac{4 \sin^2 \theta}{(1 + \cos\theta)^2}$$

or

$$\cos\theta = \frac{4 - L^2 - M^2}{4 + L^2 + M^2}\,.$$

Then, solving equations 3 and 5 for $\cos\delta \cos\Delta\alpha$, equating the results, and rearranging, we find

$$\begin{aligned} \delta &= \sin^{-1}\left(\cos\theta \sin\delta_0 + M \cos\delta_0\,(1 + \cos\theta)/2\right) \\ \alpha &= \alpha_0 + \sin^{-1}\left(\frac{L\,(1 + \cos\theta)}{2\cos\delta}\right)\,. \end{aligned}$$

We do have to test to make sure that the correct value of M is given by the chosen root of the $\sin^{-1}$ in the computation for α.

2.1.3 Find α, y from x, δ

Using equations 1 for x and substituting L and M from equations 7, we find an equation for $\Delta\alpha$ of the form

$$A \sin\Delta\alpha - B \cos\Delta\alpha = C\,,$$

where

$$\begin{aligned} A &\equiv 2\cos\delta\cos\rho \\ B &\equiv 2\cos\delta\sin\delta_0\sin\rho + x\cos\delta\cos\delta_0 \\ C &\equiv x + x\sin\delta\sin\delta_0 - 2\sin\delta\cos\delta_0\sin\rho\,. \end{aligned}$$

This is simply an equation for the sin of the difference of two angles and has solution

$$\alpha = \alpha_0 + \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{C}{\sqrt{A^2 + B^2}}\right)\,,$$

which then allows us to compute

$$y = 2\,\frac{(\sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha)\cos\rho - \cos\delta\sin\Delta\alpha\sin\rho}{1 + \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha}\,.$$

2.1.4 Find x, δ from α, y

Reversing the equation for y above, we get another equation of the form

$$A \sin\delta - B \cos\delta = C\,,$$

where

$$\begin{aligned} A &\equiv 2\cos\delta_0\cos\rho - y\sin\delta_0 \\ B &\equiv (y\cos\delta_0 + 2\sin\delta_0\cos\rho)\cos\Delta\alpha + 2\sin\rho\sin\Delta\alpha \\ C &\equiv y\,. \end{aligned}$$

This is simply an equation for the sin of the sum of two angles and has solution

$$\delta = \tan^{-1}\left(\frac{B}{A}\right) + \sin^{-1}\left(\frac{C}{\sqrt{A^2 + B^2}}\right)\,,$$

which then allows us to compute

$$x = 2\,\frac{(\sin\delta\cos\delta_0 - \cos\delta\sin\delta_0\cos\Delta\alpha)\sin\rho + \cos\delta\sin\Delta\alpha\cos\rho}{1 + \sin\delta\sin\delta_0 + \cos\delta\cos\delta_0\cos\Delta\alpha}\,.$$

2.2 AIT geometry[1]

2.2.1 Find x, y from α, δ

The IRAS documentation gives the following definition for the Aitoff coordinate system:

$$\begin{aligned}
\cos\phi &= \cos\delta\cos(\Delta\alpha/2)\\
\sin\theta &= \frac{\cos\delta\sin(\Delta\alpha/2)}{\sin\phi}\\
L &= 2\sin(\phi/2)\sin\theta\\
M &= \sin(\phi/2)\cos\theta\,.
\end{aligned}$$

This definition is difficult to use because of the considerable non-linearity and because the sign of ϕ, which is important, remains undefined. An alternate definition, which may be shown to be equivalent at $\delta_0 = 0$ within a scale factor, is given by

$$\begin{aligned}
L &= \frac{2f_\alpha\cos\delta\sin(\Delta\alpha/2)}{Z}\\
M &= \frac{f_\delta\sin\delta}{Z} - M_0\,,
\end{aligned} \tag{8}$$

where Z and M_0 are defined, for convenience, by

$$\begin{aligned}
Z &= \sqrt{\frac{1+\cos\delta\cos(\Delta\alpha/2)}{2}}\\
M_0 &= \frac{f_\delta\sin\delta_0}{\sqrt{(1+\cos\delta_0)/2}}\,.
\end{aligned}$$

Equations 1 may then be used to find x and y.

The scaling factors f_α and f_δ are a generalization of the geometry to allow the specification of a "reference pixel" at some latitude other than zero, *i.e.*, inside the field of view. The axis increments given are, by convention, arc seconds on the sky evaluated at the reference pixel. The linear system (x, y) is derived from the pixel positions (i, j) in this convention by

$$\begin{aligned}
x &= \Delta_x(i - i_0)\\
y &= \Delta_y(j - j_0)\,,
\end{aligned}$$

where the Δs are the axis increments and (i_0, j_0) is the reference pixel position. To handle rotations, we define

$$\begin{aligned}
\Delta_\alpha &= \Delta_x\cos\rho - \Delta_y\sin\rho\\
\Delta_\delta &= \Delta_y\cos\rho + \Delta_x\sin\rho\,.
\end{aligned}$$

Then the scaling parameters for this geometry are given by

$$\begin{aligned}
f_\alpha &= \frac{\Delta_\alpha\sqrt{\left(1+\cos\delta_0\cos(\frac{\Delta_\alpha}{2})\right)/2}}{2\cos\delta_0\sin(\frac{\Delta_\alpha}{2})}\\
f_\delta &= \frac{\Delta_\delta}{\frac{\sin(\delta_0+\Delta_\delta)}{\sqrt{(1+\cos(\delta_0+\Delta_\delta))/2}} - \frac{\sin\delta_0}{\sqrt{(1+\cos\delta_0)/2}}}\,.
\end{aligned}$$

Note that the choice of $\delta_0 = 0$ makes f_α and f_δ very close to one, as one would expect.

[1] **1993:** The new proposal provides an entirely new generalization to non-zero reference pixels. These equations match those of the new proposal only when $\alpha_0 = \delta_0 = M_0 = 0$ and $f_\alpha = f_\delta = 1$.

2.2.2 Find α, δ from x, y

Given x, y, we use equations 2 to derive L, M. It turns out that Z may then be computed from L and M as follows:

$$
\begin{aligned}
Z &\equiv \sqrt{\frac{1+\cos\delta\cos(\Delta\alpha/2)}{2}} \\
Z &= \frac{1}{2}\sqrt{\frac{1+2\cos\delta\cos(\Delta\alpha/2)+\cos^2\delta\cos^2(\Delta\alpha/2)}{(1+\cos\delta\cos(\Delta\alpha/2))/2}} \\
Z &= \frac{1}{2}\sqrt{\frac{4Z^2-1+\cos^2\delta\cos^2(\Delta\alpha/2)}{Z^2}} \\
Z &= \frac{1}{2}\sqrt{4-\frac{\cos^2\delta+\sin^2\delta-\cos^2\delta\cos^2(\Delta\alpha/2)}{Z^2}} \\
Z &= \frac{1}{2}\sqrt{4-\left(\frac{L}{2f_\alpha}\right)^2-\left(\frac{M+M_0}{f_\delta}\right)^2}
\end{aligned}
$$

using equations 8. Then,

$$
\begin{aligned}
\delta &= \sin^{-1}\left(\frac{(M+M_0)Z}{f_\delta}\right) &(9)\\
\alpha &= \alpha_0 + 2\sin^{-1}\left(\frac{LZ}{2f_\alpha\cos\delta}\right) &(10)
\end{aligned}
$$

follow directly.

2.2.3 Find α, y from x, δ

Like the ARC geometry, the AIT geometry requires iterative methods for two of the four problems. Some of the unknowns appear primarily in Z which is only weakly dependent on α and δ. However, the iterative methods do encounter a variety of problems in more extreme, but reasonably normal, cases. The iterative methods described in this memo are meant to be illustrative rather than an exact description of the methods implemented in $\mathcal{AIPS}$. The subroutines themselves employ derivatives of the functions with various restraining and bounding conditions to estimate the unknown parameters for the next iteration, *i.e.*, a bounded version of Newton's method.

Equations 2 and 8, after a bit of rearrangement yield

$$
\begin{aligned}
y &= \frac{f_\delta\sin\delta}{Z\cos\rho}-\frac{x\sin\rho}{\cos\rho}-\frac{M_0}{\cos\rho} \\
\sin(\Delta\alpha/2) &= \left(\frac{Z}{2f_\alpha\cos\delta}\right)(x\cos\rho - y\sin\rho)
\end{aligned}
$$

or

$$
\sin(\Delta\alpha/2) = \frac{xZ+(M_0Z-f_\delta\sin\delta)\sin\rho}{2f_\alpha\cos\delta\cos\rho}. \qquad (11)
$$

The iterative method simply involves guessing $(\Delta\alpha/2)$ and then looping between computing Z and recomputing $(\Delta\alpha/2)$ using equation 11. When convergence is achieved, y may be computed with the equation given above.

2.2.4 Find x, δ from α, y

Similarly, equations 2 and 8, after a bit of rearrangement yield

$$\begin{aligned} x &= \frac{2f_\alpha \sin(\Delta\alpha/2)\cos\delta}{Z\cos\rho} + \frac{y\sin\rho}{\cos\rho} \\ \sin\delta &= \frac{Z}{f_\delta}\left(y\cos\rho + x\sin\rho + M_0\right) \end{aligned}$$

or

$$\sin\delta = \frac{yZ + M_0 Z\cos\rho + 2f_\alpha \cos\delta\sin\rho\sin(\Delta\alpha/2)}{f_\delta\cos\rho}. \tag{12}$$

The iterative method simply involves guessing δ and then looping between computing Z and recomputing δ using equation 12. When convergence is achieved, x may be computed with the equation given above.

2.3 GLS geometry[2]

2.3.1 Find x, y from α, δ

The "global sinusoidal" coordinates are defined by

$$\begin{aligned} L &= \Delta\alpha\cos\delta \\ M &= \Delta\delta \end{aligned} \tag{13}$$

and x and y may be determined by equations 1.

2.3.2 Find α, δ from x, y

Using equations 2 and 13,

$$\delta = M + \delta_0$$

or

$$\delta = \delta_0 + x\sin\rho + y\cos\rho.$$

Then

$$\alpha = \alpha_0 + \frac{L}{\cos\delta}$$

or

$$\alpha = \alpha_0 + \frac{x\cos\rho - y\sin\rho}{\cos\delta}.$$

2.3.3 Find α, y from x, δ

Again using equations 2 and 13,

$$y = \frac{\Delta\delta - x\sin\rho}{\cos\rho}$$

and then

$$\alpha = \alpha_0 + \frac{x\cos\rho - y\sin\rho}{\cos\delta}.$$

[2] **1993:** The generalization to reference pixels other than $(\alpha_0, \delta_0) = (0, 0)$ is entirely different in the FITS proposal. Older images may be corrected simply by changing the declination reference value to zero with a corresponding linear shift of the reference pixel location.

2.3.4 Find x, δ from α, y

The last of the problems does not yield a simple, analytic solution. We proceed as follows, using equations 1, 2 and 13 to get

$$\begin{aligned} \delta &= \delta_0 + x\sin\rho + y\cos\rho \\ x &= \Delta\alpha\cos\delta\cos\rho + \Delta\delta\sin\rho, \end{aligned}$$

which yields

$$\delta = \delta_0 + \frac{y + \Delta\alpha\cos\delta\sin\rho}{\cos\rho}\,.$$

An iterative method may be used, beginning with $\delta = \delta_0 + y/\cos\rho$, to derive eventually a correct value for δ. Then the equation above for x may be applied.

2.4 MER geometry[3]

2.4.1 Find x, y from α, δ

In the Mercator projection, lines of longitude are parallel and evenly spaced while lines of latitude are parallel with a spacing which increases as one approaches the pole. The basic equations are

$$\begin{aligned} L &= f_\alpha\,\Delta\alpha \\ M &= f_\delta\,\ln\left(\tan(\frac{\delta}{2} + \frac{\pi}{4})\right) - M_0\,, \end{aligned} \tag{14}$$

with

$$\begin{aligned} f_\alpha &= \cos\delta_0 \\ f_\delta &= \frac{\Delta_\delta}{\ln\left(\tan(\frac{\delta_0+\Delta_\delta}{2} + \frac{\pi}{4})\right) - \ln\left(\tan(\frac{\delta_0}{2} + \frac{\pi}{4})\right)} \\ M_0 &= f_\delta\,\ln\left(\tan(\frac{\delta_0}{2} + \frac{\pi}{4})\right) \end{aligned}$$

providing a service similar to that provided in the AIT geometry. The peculiar form of this geometry comes from from integrating $\sec\delta$.

2.4.2 Find α, δ from x, y

(L, M) are found from (x, y) by equations 2. Then

$$\begin{aligned} \alpha &= L/f_\alpha + \alpha_0 \\ \delta &= 2\tan^{-1}\left(e^{(\frac{M+M_0}{f_\delta})}\right) - \pi/2\,. \end{aligned}$$

2.4.3 Find α, y from x, δ

Equations 1 and 14 are manipulated as follows:

$$M = f_\delta\,\ln\left(\tan(\frac{\delta}{2} + \frac{\pi}{4})\right) - M_0$$

[3] **1993:** The new proposal provides an entirely new generalization to non-zero reference pixels. These equations match those of the new proposal only when $\alpha_0 = \delta_0 = M_0 = 0$ and $f_\alpha = f_\delta = 1$.

$$\begin{aligned} L &= \frac{x - M\sin\rho}{\cos\rho} \\ y &= M\cos\rho - L\sin\rho \\ \alpha &= \alpha_0 + L/f_\alpha\,. \end{aligned}$$

2.4.4 Find x, δ from α, y

Similarly, equations 1 and 14 may be manipulated as follows:

$$\begin{aligned} L &= f_\alpha\,\Delta\alpha \\ M &= \frac{y + L\sin\rho}{\cos\rho} \\ \delta &= 2\tan^{-1}\left(e^{(\frac{M+M_0}{f_\delta})}\right) - \pi/2 \\ x &= L\cos\rho + M\sin\rho\,. \end{aligned}$$

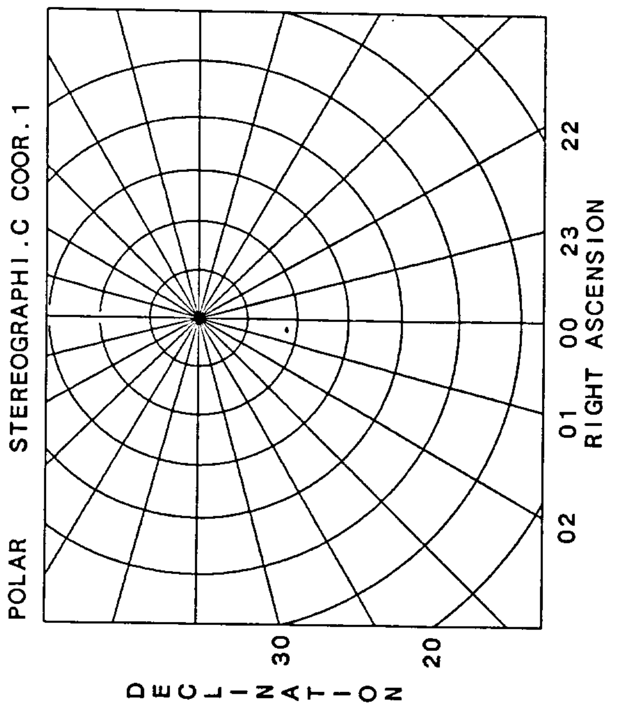
3 *AIPS* Implementation

These new coordinates are implemented in *AIPS* with the same subroutines and commons which are used for the previously documented geometries and, hence, their introduction should be relatively transparent to most programmers. The subroutine `SETLOC` recognizes the new coordinates by the suffixes `-STG`, `-GLS`, `-AIT`, and `-MER` in the axis type parameter of the image header. The `AXFUNC` parameter in common `/LOCATI/` is then set to 9, 6, 8, or 7, respectively. The subroutines `DIRCOS`, `NEWPOS`, `DIRRA` and `DIRDEC` apply the geometry to solve the four problems posed here. In these geometries, there can be large regions in a rectangular image which are forbidden, *i.e.*, which have no latitude and longitude. To handle this condition properly, error return parameters were added to the call sequences of the basic position routines and a new subroutine, `TICINC`, was written to handle the more elaborate steps now used to determine tick values and increments. Constants for use by each geometry were added to the location common and are initialized by `SETLOC`. They are f_α = `GEOMD1`, f_δ = `GEOMD2`, and M_0 = `GEOMD3` in the MER and AIT geometries.

The global shapes of the non-projective geometries are illustrated on the next page of this memo. These figures were produced with *AIPS* using the new `DOCIRCLE` option designed to show full coordinate grids.

4 Acknowledgements

The STG geometry was suggested by Don Wells; the GLS geometry was suggested and documented by Jim Condon; the formulation of the AIT geometry used here was given to me by Don Wells; and the basics of the MER geometry were taken from W. M. Smart, *Text-Book on Spherical Astronomy.* The IRAS formulation, including the "trick" used in section 2.2.2, is contained in *IRAS Catalogs and Atlases, Explanatory Supplement*, Birchman, C. A., Neugebauer, G., Habing, H. J., Clegg, P. E., and Chester, T. J., `JPL D--1855`. Don Wells also made very useful suggestions on this manuscript and on the iterative methods actually employed. The National Radio Astronomy Observatory is operated by Associated Universities Inc. under contract with the National Science Foundation.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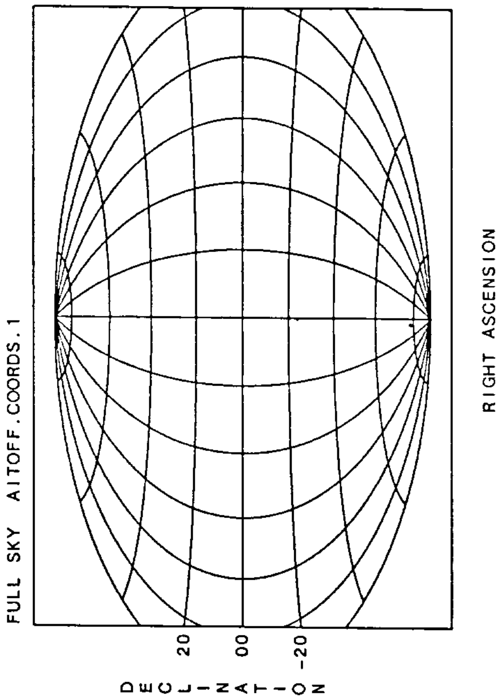
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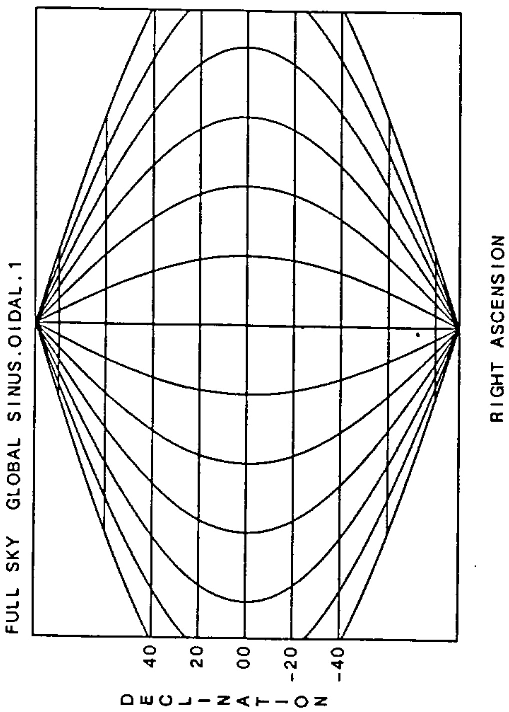
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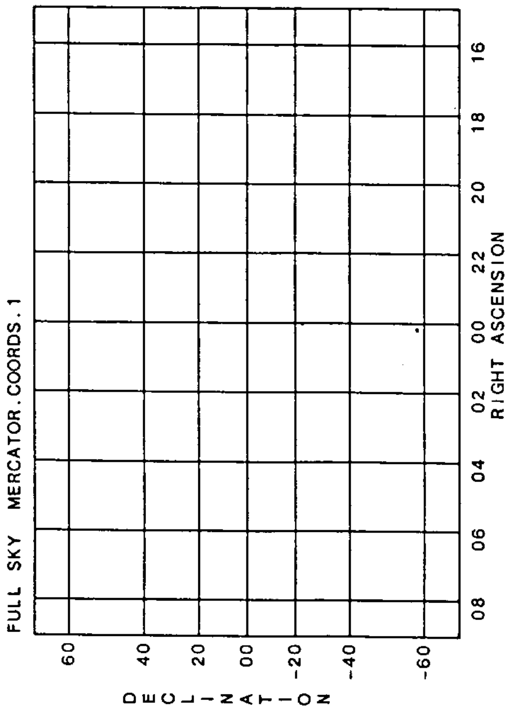
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